

A TYPE OF OSCILLATION WITHIN THE HELIUM ATOM

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A TYPE OF OSCILLATION WITHIN THE HELIUM ATOM

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Introduction. In a recent paper* Professor H. E. Buchanan works out the small oscillations of the electrons near the equilateral triangle positions, considering the electrical forces only. His article is one of a rather large number that have been published recently on the subject of the neutral helium atom.†

* American Mathematical Monthly, vol. 38 (1931), pp. 511-521.

† Rawles, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 618-630; Van Vleck, Bulletin of the National Research Council, vol. 10, part 4, (1926), Chapter 7; D. Buchanan, Transactions of the Royal Society of Canada, Series 3, vol. 23 (1929), pp. 227-245; U. Crudeli, Rendiconti Del Circolo Matematico Di Palermo, vol. 51 (1926), pp. 1-20; Langmuir, Physical Review, vol. 17 (1921), pp. 339-353.

This paper gives the variations of Professor Buchanan's solution when both gravitational and electrical forces are taken into account.

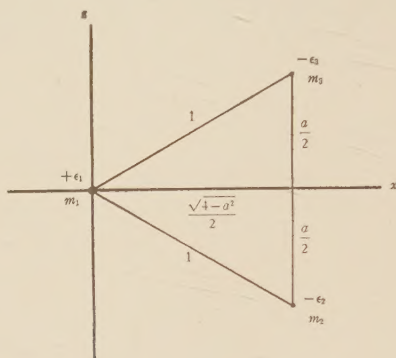


FIG. 1

Given the positive nucleus ϵ_1 with mass m_1 and the two electrons with masses m_2 and m_3 and charges $-\epsilon_2$ and $-\epsilon_3$ respectively, as shown in Fig. 1. Assuming Coulomb's Law for the forces due to electrical charges, and Newton's Law for the gravitational attraction, the relation $Ma = F$ gives the differential equation:

$$(1) \quad \begin{aligned} m_i \frac{d^2 x_i}{dt^2} &= \frac{\partial U}{\partial x_i}, \\ m_i \frac{d^2 y_i}{dt^2} &= \frac{\partial U}{\partial y_i}, \\ m_i \frac{d^2 z_i}{dt^2} &= \frac{\partial U}{\partial z_i}, \end{aligned} \quad i = 1, 2, 3;$$

where

$$U = \frac{k^2 \epsilon_1 \epsilon_2}{r_{12}} + \frac{k^2 \epsilon_1 \epsilon_3}{r_{13}} - \frac{k^2 \epsilon_2 \epsilon_3}{r_{23}} + \frac{K^2 m_1 m_2}{r_{12}} + \frac{K^2 m_2 m_3}{r_{23}} + \frac{K^2 m_3 m_1}{r_{31}},$$

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}, \quad i, j = 1, 2, 3, \quad j \neq i;$$

and where

k^2 proportionality factor in Coulomb's Law,
 K^2 gravitational constant.

Let the motion of the bodies be referred to a new system of axes having the same origin as the old, and rotating in the $\xi\eta$ -plane in the direction in which the electrons move with uniform angular velocity ω .

Substituting

$$\begin{aligned} x_i &= \xi_i \cos \omega t - \eta_i \sin \omega t, \\ y_i &= \xi_i \sin \omega t + \eta_i \cos \omega t, \\ z_i &= \zeta_i, \end{aligned} \quad i = 1, 2, 3,$$

in (1), the equations become

$$(2) \quad \begin{aligned} \frac{d^2\xi_i}{dt^2} - 2\omega \frac{d\eta_i}{dt} - \omega^2\xi_i - \frac{1}{m_i} \frac{\partial U}{\partial \xi_i} &= 0, \\ \frac{d^2\eta_i}{dt^2} + 2\omega \frac{d\xi_i}{dt} - \omega^2\eta_i - \frac{1}{m_i} \frac{\partial U}{\partial \eta_i} &= 0, \\ \frac{d^2\zeta_i}{dt^2} - \frac{1}{m_i} \frac{\partial U}{\partial \zeta_i} &= 0, \end{aligned} \quad i = 1, 2, 3.$$

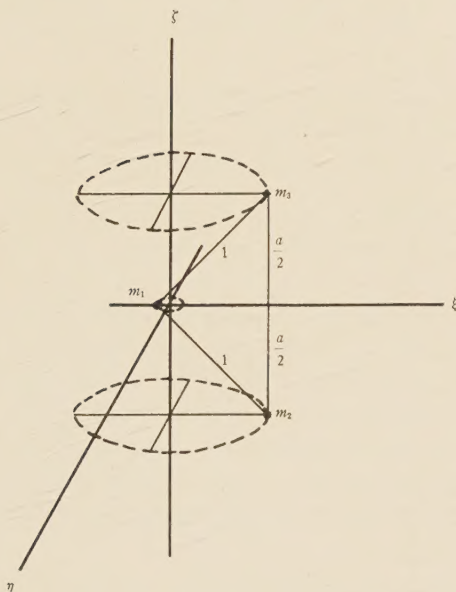


FIG. 2

Triangular Solutions. We wish a solution of equations (2) in which the bodies remain at the vertices of a triangle and rotate in circles, as shown in Fig. 2. ξ_i , η_i , ζ_i are constants and their derivatives are zero. We choose $r_{13} = r_{12} = 1$, $r_{32} = a$. A solution will exist if (2) can be satisfied, that is, if we can have simultaneously

$$(3) \quad \begin{aligned} &-m_1\omega^2\xi_1 + k^2\epsilon_1\epsilon_2(\xi_1 - \xi_2) \\ &+ k^2\epsilon_1\epsilon_3(\xi_1 - \xi_3) + K^2m_1m_2(\xi_1 - \xi_2) + K^2m_1m_3(\xi_1 - \xi_3) = 0, \\ &-m_2\omega^2\xi_2 + k^2\epsilon_1\epsilon_2(\xi_2 - \xi_1) \\ &-\frac{k^2}{a^3}\epsilon_2\epsilon_3(\xi_2 - \xi_3) + K^2m_1m_2(\xi_2 - \xi_1) + \frac{K^2}{a^3}m_2m_3(\xi_2 - \xi_3) = 0, \\ &-m_3\omega^2\xi_3 + k^2\epsilon_1\epsilon_3(\xi_3 - \xi_1) \\ &-\frac{k^2}{a^3}\epsilon_2\epsilon_3(\xi_3 - \xi_2) + K^2m_1m_3(\xi_3 - \xi_1) + \frac{K^2}{a^3}m_3m_2(\xi_3 - \xi_2) = 0, \end{aligned}$$

$$\begin{aligned}
 & -m_1\omega^2n_1 + k^2\epsilon_1\epsilon_2(\eta_1 - \eta_2) \\
 & + k^2\epsilon_1\epsilon_3(\eta_1 - \eta_3) + K^2m_1m_2(\eta_1 - \eta_2) + K^2m_1m_3(\eta_1 - \eta_3) = 0, \\
 & -m_2\omega^2\eta_2 + k^2\epsilon_1\epsilon_2(\eta_2 - \eta_1) \\
 (4) \quad & -\frac{k^2\epsilon_2\epsilon_3}{a^3}(\eta_2 - \eta_3) + K^2m_1m_2(\eta_2 - \eta_1) + \frac{K^2m_2m_3}{a^3}(\eta_2 - \eta_3) = 0, \\
 & -m_3\omega^2\eta_3 + k^2\epsilon_1\epsilon_3(\eta_3 - \eta_1) \\
 & -\frac{k^2\epsilon_2\epsilon_3}{a^3}(\eta_3 - \eta_2) + K^2m_1m_3(\eta_3 - \eta_1) + \frac{K^2m_2m_3}{a^3}(\eta_3 - \eta_2) = 0,
 \end{aligned}$$

$$\begin{aligned}
 & k^2\epsilon_1\epsilon_2(\zeta_1 - \zeta_2) + k^2\epsilon_1\epsilon_3(\zeta_1 - \zeta_3) + K^2m_1m_2(\zeta_1 - \zeta_2) + K^2m_1m_3(\zeta_1 - \zeta_3) = 0, \\
 (5) \quad & k^2\epsilon_2\epsilon_1(\zeta_2 - \zeta_1) - \frac{k^2\epsilon_2\epsilon_3}{a^3}(\zeta_2 - \zeta_3) + K^2m_2m_1(\zeta_2 - \zeta_1) + \frac{K^2m_2m_3}{a^3}(\zeta_2 - \zeta_3) = 0, \\
 & k^2\epsilon_3\epsilon_1(\zeta_3 - \zeta_1) - \frac{k^2\epsilon_3\epsilon_2}{a^3}(\zeta_3 - \zeta_2) + K^2m_3m_1(\zeta_3 - \zeta_1) + \frac{K^2m_3m_2}{a^3}(\zeta_3 - \zeta_2) = 0.
 \end{aligned}$$

In order that (3), (4), and (5) have a solution other than trivial ones it is necessary that each of the three third order determinants composed respectively of the coefficients of $\xi_i, \eta_i, \zeta_i, i = 1, 2, 3$, should vanish. The determinant of coefficients of ζ_i does not contain ω and vanishes identically. The determinant of coefficients of η_i is the same as that of the coefficients of ξ_i . We equate this determinant to zero, and let $m_2 = m_3 = m$, $2\epsilon = \epsilon_1 = 2\epsilon_2 = 2\epsilon_3$. After some reductions, $-\omega^2$ appears as a common factor of one of the rows. Removing it, the equation becomes

$$\begin{vmatrix}
 m_1 & -4k^2\epsilon^2 - 2m_1mK^2 & 0 \\
 m & -m\omega^2 + 2k^2\epsilon^2 + m_1mK^2 & \frac{m\omega^2}{2} - k^2\left(\epsilon^2 - \frac{\epsilon^2}{a^3}\right) - K^2\left(\frac{m_1m}{2} + \frac{m^2}{a^3}\right) \\
 m & -m\omega^2 + 2k^2\epsilon^2 + m_1mK^2 & -\frac{m\omega^2}{2} + k^2\left(\epsilon^2 - \frac{\epsilon^2}{a^3}\right) + K^2\left(\frac{m_1m}{2} + \frac{m^2}{a^3}\right)
 \end{vmatrix} = 0$$

which may be written as the product of two factors, and either one may be equal to zero. Thus

$$(6a) \quad \frac{m\omega^2}{2} - k^2\left(\epsilon^2 - \frac{\epsilon^2}{a^3}\right) - K^2\left(\frac{m_1m}{2} + \frac{m^2}{a^3}\right) = 0,$$

or

$$(6b) \quad \begin{vmatrix}
 m_1 & -4k^2\epsilon^2 - 2m_1mK^2 & 0 \\
 m & -m\omega^2 + 2k^2\epsilon^2 + K^2m_1m & 1 \\
 m & -m\omega^2 + 2k^2\epsilon^2 + K^2m_1m & 1
 \end{vmatrix} = 0.$$

From the first factor

$$(7) \quad \omega = \pm \sqrt{\frac{2k^2\epsilon^2}{m}\left(1 - \frac{1}{a^3}\right) + 2K^2\left(\frac{m_1}{2} + \frac{m}{a^3}\right)}.$$

This will be a solution of the problem, i.e. a possible value of the velocity of rotation, as long as the expression inside the radical is positive. The plus or minus sign in front of the radical indicates rotation in one or the other direction; clockwise or counter-clockwise. (If $K=0$ and $a=1$, this reduces to zero, one of the values found by Professor Buchanan.)

Hence

$$(7a) \quad \begin{aligned} &\frac{2k^2\epsilon^2}{m}(a^3 - 1) + K^2(m_1a^3 + 2m) \geq 0 \\ &\left(\frac{2k^2\epsilon^2}{m} + m_1K^2\right)a^3 > \frac{2k^2\epsilon^2}{m} - 2K^2m \\ &a^3 \geq \frac{2k^2\epsilon^2 - 2K^2m^2}{2k^2\epsilon^2 + K^2mm_1}. \end{aligned}$$

Now

$$\epsilon = 4.77 \times 10^{-10} \text{ e.s.u.}$$

$$k = 1$$

$$K = 6.69 \times 10^{-8}$$

$$m = 8.99 \times 10^{-28} \text{ g.}$$

$$m_1 = 6.62 \times 10^{-24} \text{ g.}$$

Substituting these values in the right-hand side of the inequality it is seen that the second member of both numerator and denominator has practically no influence on the value of the fraction. The right-hand side is therefore approximately unity. The conclusion is that a must be greater than a value slightly less than one, otherwise the expression inside the radical in (7) will be negative, making the velocity imaginary, and hence there would be no solution.

Now consider the second factor, namely, (6b). Add the third row to the second, then expand the determinant and solve for ω . We have

$$(8) \quad \omega = \pm \sqrt{\frac{(2k^2\epsilon^2 + m_1mK^2)(m_1 + 2m)}{mm_1}}$$

which, if K is put equal to zero, reduces to the value of ω found by Professor Buchanan.

We consider first the variations from the equilateral triangle positions due to the gravitational forces. If in (3) we put $K^2=0$ and $a=1$, ξ_1 and ξ_2 can be found in terms of ξ_3 . There results $\xi_2 = \xi_3$, $\xi_1 = -2m\xi_3/m_1$. The values of η_1 , η_2

and η_3 could be found in the same way from equations (4), but it would be easier to satisfy (4) by $\eta_1 = \eta_2 = \eta_3 = 0$. Equations (5) are satisfied by $\zeta_1 = 0$, $\zeta_2 = -\zeta_3 = \frac{1}{2}$. The motion is then as shown in the second figure. The equilateral triangle is in the $\xi\zeta$ -plane, and rotates about the ζ axis. The electrons and the nucleus move in circles whose planes are parallel to the $\xi\eta$ -plane.

The equations for small oscillations. In (2) let

$$\begin{aligned}\xi_i &= \xi_i^{(0)} + x_i, \\ \eta_i &= y_i, \\ \zeta_i &= \zeta_i^{(0)} + z_i,\end{aligned}\quad i = 1, 2, 3,$$

where x_i , y_i , and z_i represent the variations from the equilateral triangle solutions $\xi_i^{(0)}$, 0, $\zeta_i^{(0)}$. By means of the center of gravity equations

$$m_1x_1 + m(x_2 + x_3) = 0, \quad m_1y_1 + m(y_2 + y_3), \quad m_1z_1 + m(z_2 + z_3) = 0,$$

x_2 , y_2 , and z_2 are eliminated. Equations (2) then become

$$\begin{aligned}(9) \quad & \frac{d^2x_1}{dt^2} - 2\omega \frac{dy_1}{dt} = \omega^2(\xi_1^{(0)} + x_1) \\ & - \frac{H}{m_1} \left(\frac{\xi_1^{(0)} + x_1 - \xi_2^{(0)} - x_2}{r_{12}^3} + \frac{\xi_1^{(0)} + x_1 - \xi_3^{(0)} - x_3}{r_{13}^3} \right) \\ & \frac{d^2x_3}{dt^2} - 2\omega \frac{dy_3}{dt} = \omega^2(\xi_3^{(0)} + x_3) \\ & - \frac{H}{m} \left(\frac{\xi_3^{(0)} + x_3 - \xi_1^{(0)} - x_1}{r_{31}^3} \right) + \frac{J}{m} \left(\frac{\xi_3^{(0)} + x_3 - \xi_2^{(0)} - x_2}{r_{32}^3} \right) \\ & \frac{d^2y_1}{dt^2} + 2\omega \frac{dx_1}{dt} = \omega^2y_1 - \frac{H}{m_1} \left(\frac{y_1 - y_2}{r_{12}^3} + \frac{y_1 - y_3}{r_{13}^3} \right) \\ & \frac{d^2y_3}{dt^2} + 2\omega \frac{dx_3}{dt} = \omega^2y_3 - \frac{H}{m} \left(\frac{y_3 - y_1}{r_{31}^3} \right) + \frac{J}{m} \left(\frac{y_3 - y_2}{r_{32}^3} \right) \\ & \frac{d^2z_1}{dt^2} = - \frac{H}{m_1} \left(\frac{z_1 - \zeta_2^{(0)} - z_2}{r_{12}^3} + \frac{z_1 - \zeta_3^{(0)} - z_3}{r_{13}^3} \right) \\ & \frac{d^2z_3}{dt^2} = - \frac{H}{m} \left(\frac{\zeta_3^{(0)} + z_3 - z_1}{r_{31}^3} \right) + \frac{J}{m} \left(\frac{\zeta_3^{(0)} + z_3 - \zeta_2^{(0)} - z_2}{r_{32}^3} \right)\end{aligned}$$

where

$$H = 2k^2\epsilon^2 + m_1mK^2, \quad J = k^2\epsilon^2 - m^2K^2.$$

We develop the right-hand side of (9) into power series in x_i , y_i , z_i , $i=1, 3$, by means of Taylor's Formula. Considering only the linear terms in these series we have, on expanding about the triangular points,

$$\begin{aligned}
 (10) \quad & \frac{d^2 x_1}{dt^2} - 2\omega \frac{dy_1}{dt} = A_{11}x_1 + C_{11}z_1 + C_{13}z_3 \\
 & \frac{d^2 x_3}{dt^2} - 2\omega \frac{dy_3}{dt} = A_{21}x_1 + A_{23}x_3 + C_{21}z_1 + C_{23}z_3 \\
 & \frac{d^2 y_1}{dt^2} + 2\omega \frac{dx_1}{dt} = 0 \\
 & \frac{d^2 y_3}{dt^2} + 2\omega \frac{dx_3}{dt} = B_{41}y_1 + B_{43}y_3 \\
 & \frac{d^2 z_1}{dt^2} = A_{51}x_1 + A_{53}x_3 + C_{51}z_1 \\
 & \frac{d^2 z_3}{dt^2} = A_{61}x_1 + A_{63}x_3 + C_{61}z_1 + C_{63}z_3,
 \end{aligned}$$

where

$$\begin{aligned}
 A_{11} &= \omega^2 \left(3 - \frac{3a^2}{4} \right), \quad C_{11} = -\frac{H}{m} \frac{3a\sqrt{4-a^2}}{4}, \quad C_{13} = -\frac{H}{m_1} \frac{3a\sqrt{4-a^2}}{2}, \\
 A_{21} &= -\frac{H}{m} \left(2 - \frac{3a^2}{4} \right) - J \frac{m_1}{a^3 m^2}, \quad A_{23} = \omega^2 + \frac{H}{m} \left(\frac{8-3a^2}{4} \right) + \frac{2J}{ma^3}, \\
 C_{21} &= -\frac{H}{m} \frac{3a\sqrt{4-a^2}}{4}, \quad C_{23} = \frac{H}{m} \frac{3a\sqrt{4-a^2}}{4}, \quad B_{41} = \frac{H}{m} + \frac{m_1 J}{m^2 a^3}, \\
 B_{43} &= \omega^2 - \frac{H}{m} + \frac{2J}{ma^3}, \quad A_{51} = -H \frac{3a}{4m} \sqrt{4-a^2}, \quad A_{53} = -\frac{H}{m_1} \frac{3a}{2} \sqrt{4-a^2}, \\
 C_{51} &= -\frac{H}{m_1} \frac{4-3a^2}{4} \left(\frac{2m+m_1}{m} \right), \quad A_{61} = -\frac{H}{m} \frac{3a}{4} \sqrt{4-a^2}, \\
 A_{63} &= \frac{H}{m} \frac{3a}{4} \sqrt{4-a^2}, \quad C_{61} = \frac{H}{m} \left(1 - \frac{3a^2}{4} \right) - \frac{2m_1 J}{a^3 m^2}, \\
 C_{63} &= -\frac{H}{m} \left(1 - \frac{3a^2}{4} \right) - \frac{4J}{a^3 m}.
 \end{aligned}$$

The characteristic equations. In (10) let

$$x_i = K_i e^{\lambda t}, \quad y_i = L_i e^{\lambda t}, \quad z_i = M_i e^{\lambda t}, \quad i = 1, 3,$$

to obtain the solution of the system for the values of ω given in (7) and (8). This substitution gives six linear equations in K_i, L_i, M_i with the common factor $e^{\lambda t}$, which may be divided out. In order that there be a solution other than $K_i = L_i = M_i = 0, i = 1, 3$, the determinant formed of the coefficients of $K_i, L_i,$

M_i in these equations should equal zero. This determinant is of the sixth order in λ^2 . It may be simplified by dividing out λ from the third row, and making the following substitutions

ω^2 as given in (8)

$$\lambda = \sqrt{\frac{H}{2}} x, \quad s = \frac{2J}{mH}.$$

After some reductions x may be divided out, leaving a fifth order determinant in x^2 . The value of a is set equal to 1, the unit of mass is chosen so that $m = 1$, and $1/m_1$ set equal to y . The determinant is then expanded to give the following equation in x

$$\begin{aligned} & x^{10} + 8(2y + 1)x^8 + \left(6y^2 + \frac{105}{2}y + 6 - 108y^3 + 48ys + 21s - 12s^2\right)x^6 \\ & + \left(604y^3 + 324y^2 - 8y + \frac{19}{2} + 432y^4 + 610sy^2 + 386sy + \frac{957s}{2} - 10s^2\right. \\ & \left.+ 16s^3 + 216y^3s - 32s^2y\right)x^4 + \left(112y^4 - 48y^3 - 424y^2 - 298y - 56\right. \\ (11) \quad & \left.+ 904sy^3 + 541sy^2 + \frac{413}{2}sy + 81s + 51s^2 + 64s^3 + 102s^2y + 128s^3y\right)x^2 \\ & + (28s^3 + 242s^2 - 36s - 160y^4 - 988y^3 - 868y^2 - 207y - 428sy \\ & + 1080s^2y + 112s^3y + 1416s^2y^2 + 2256y^2s + 2736sy^3 + 448sy^4 \\ & + 448s^2y^3 + 112y^2s^3) = 0. \end{aligned}$$

If now we put $K=0$, the parameter s reduces to 1 and the above equation reduces to a much simpler equation of the tenth degree, agreeing with the one found by Professor Buchanan. He found two roots of this equation to be $x = \pm iu$, so dividing thru by the factor $x^2 + u^2$, he secured the following eighth degree equation, where y has the value .00014,

$$\begin{aligned} & x^8 + 6(2y + 1)x^6 + 3(13y^2 + 22y + 1)x^4 + (2y + 1)^2(7y + 62)x^2 \\ & + 18(2y + 1)^2(y + 5) = 0. \end{aligned}$$

He found the following for roots of this equation

$$\begin{aligned} x^2 &= .99965 \dots - i3.00063 \dots, & x^2 &= .99965 \dots + i3.00063 \dots, \\ x^2 &= -1.35382 \dots, & x^2 &= -6.64717 \dots. \end{aligned}$$

The computed value of $s = -.00027$. We can get a solution of (11) in the form $x^2 = \phi + \psi s + \dots$, and since s is very small we need take only the linear terms. Substituting this value of x^2 in equation (11) and collecting the first degree terms in s we find

$$\psi = \frac{-21\phi^3 - \frac{957}{2}\phi + 36}{5\phi^4 + 64y\phi^2 + 32\phi^3 + \frac{315}{2}y\phi^2 + 18\phi^2 - 8y + \frac{19}{2}}$$

where ϕ is the value of x^2 when $s = 0$, already found above.

Therefore the approximate roots of (11) are as follows:

$$\begin{aligned} x^2 &= s3.79 \dots, & x^2 &= -2.00056 - s7.19 \dots, \\ x^2 &= .99965 \dots - i3.00063 \dots + (.759 \dots - i.251 \dots)s, \\ x^2 &= .99965 \dots + i3.00063 \dots + (.759 \dots + i.251 \dots)s, \\ x^2 &= -1.35382 \dots + s36.6 \dots, & x^2 &= -6.64717 \dots + s8.03 \dots. \end{aligned}$$

The characteristic exponents. Since s is very small, these values differ but little from the ones found by Professor Buchanan. When the numerical value of s is substituted and the square root taken, the form of the roots is as follows:

$$\begin{aligned} x &= \pm a, & x &= \pm ib, & x &= \pm (c + id), & x &= \pm (c - id) \\ x &= \pm gi, & x &= \pm hi \end{aligned}$$

where the numerical values of a, b, c, d, g , and h can be easily computed. Hence the values of λ which satisfy the original equation are not completely known. Writing ρ for $\sqrt{H/2}$, they are

$$\begin{aligned} &\pm a\rho, & &\pm ib\rho, & &\pm (c + id)\rho, \\ &\pm (c - id)\rho, & &\pm ig\rho, & &\pm ih\rho. \end{aligned}$$

The solutions of equations (10). Substitute the above values of λ in the characteristic equations

$$x_i = K_i e^{\lambda_i t}, \quad y_i = L_i e^{\lambda_i t}, \quad z_i = M_i e^{\lambda_i t}, \quad i = 1, 3$$

to obtain the general solutions of equations (10) as follows:

$$\begin{aligned} x_i &= K_{i1} e^{a\rho t} + K_{i2} e^{-a\rho t} + K_{i3} e^{ib\rho t} + K_{i4} e^{-ib\rho t} + K_{i5} e^{i\rho t} + K_{i6} e^{-i\rho t} \\ &\quad + K_{i7} e^{ih\rho t} + K_{i8} e^{-ih\rho t} + e^{c\rho t} (K_{i9} e^{id\rho t} + K_{i10} e^{-id\rho t}) \\ &\quad + e^{-c\rho t} (K_{i11} e^{id\rho t} + K_{i12} e^{-id\rho t}) \\ y_i &= L_{i1} e^{a\rho t} + L_{i2} e^{-a\rho t} + L_{i3} e^{ib\rho t} + L_{i4} e^{-ib\rho t} + L_{i5} e^{i\rho t} + L_{i6} e^{-i\rho t} \\ (12) \quad &\quad + L_{i7} e^{ih\rho t} + L_{i8} e^{-ih\rho t} + e^{c\rho t} (L_{i9} e^{id\rho t} + L_{i10} e^{-id\rho t}) \\ &\quad + e^{-c\rho t} (L_{i11} e^{id\rho t} + L_{i12} e^{-id\rho t}) \\ z_i &= M_{i1} e^{a\rho t} + M_{i2} e^{-a\rho t} + M_{i3} e^{ib\rho t} + M_{i4} e^{-ib\rho t} + M_{i5} e^{i\rho t} + M_{i6} e^{-i\rho t} \\ &\quad + M_{i7} e^{ih\rho t} + M_{i8} e^{-ih\rho t} + e^{c\rho t} (M_{i9} e^{id\rho t} + M_{i10} e^{-id\rho t}) \\ &\quad + e^{-c\rho t} (M_{i11} e^{id\rho t} + M_{i12} e^{-id\rho t}) \end{aligned}$$

where the subscript $i = 1, 3$.

These solutions consist of the non-periodic terms $e^{\pm a\rho t}$, the non-periodic vibrations $e^{(c\pm id)\rho t}$, the non-periodic damped vibrations $e^{-(c\pm id)\rho t}$ and the periodic terms $e^{\pm ib\rho t}$, $e^{\pm ig\rho t}$, $e^{\pm ih\rho t}$. So that if the system is disturbed in any way it may have periodic oscillations, but most likely will break up, because the coefficients of the non-periodic terms are such that these terms fail to cancel out.

These solutions give the variations from those of Professor Buchanan when the gravitational forces are taken into consideration. The periods of the oscillating terms are $2\pi/b\rho$, $2\pi/g\rho$, $2\pi/h\rho$.

The solutions (12) contain 72 arbitrary constants. Twelve of them can be chosen arbitrarily as constants of integration, and the others found as linear functions of these twelve.

Summary. The problem of the motion of the electrons of the helium atom as considered by Professor Buchanan has been extended to take care of gravitational forces as well as electrical forces. With some numerical changes, all of his conclusions are true in this case. An isosceles triangle solution could be found by taking $a \neq 1$, and the limitations on this solution could be considered, taking into account either gravitational forces or electrical forces, or both simultaneously.

